B. Sc. Part I (Hons.)

2nd Paper

Differential Calculus

Some impt. sums on Leibnitz's theorem

1. If $y = A\left[x+\sqrt{x^2-1}\right]^n + B\left[x-\sqrt{x^2-1}\right]^n$,

prove that

$(x^2-1)y_2 + xy_1 - n^2 y = 0$ and

$(x^2-1)y_{n+2} + (2n+1)xy_{n+1} = 0$

Soln Given that

$$y = A\left[x+\sqrt{x^2-1}\right]^n + B\left[x-\sqrt{x^2-1}\right]^n \quad (1)$$

Differentiating it with respect to x, we get

$$y_1 = nA\left[x+\sqrt{x^2-1}\right]^{n-1}\left[1+\frac{2x}{2\sqrt{x^2-1}}\right] + nB\left[x-\sqrt{x^2-1}\right]^{n-1}\left[1-\frac{2x}{2\sqrt{x^2-1}}\right]$$

$$\Rightarrow y_1 = nA\left[x+\sqrt{x^2-1}\right]^{n-1}\left[\frac{x+\sqrt{x^2-1}}{\sqrt{x^2-1}}\right] - nB\left[x-\sqrt{x^2-1}\right]^{n-1}\left[\frac{x-\sqrt{x^2-1}}{\sqrt{x^2-1}}\right]$$

$$\Rightarrow \sqrt{x^2-1}\, y_1 = nA\left[x+\sqrt{x^2-1}\right]^n - nB\left[x-\sqrt{x^2-1}\right]^n$$

squaring both sides

$$\Rightarrow (x^2-1)y_1^2 = n^2\left[A(x+\sqrt{x^2-1})^n + B(x-\sqrt{x^2-1})^n\right]^2 - 4ABn^2(x+\sqrt{x^2-1})^n(x-\sqrt{x^2-1})^n$$

$\Rightarrow (x^2-1)y_1^2 = n^2y^2 - 4ABn^2[x^2-(x^2-1)]^n$

$\Rightarrow (x^2-1)y_1^2 = n^2y^2 - 4ABn^2$

Differentiating w. r. to x, we get

$(x^2-1)\,2y_1y_2 + 2xy_1^2 = 2n^2yy_1$

Dividing by $2y_1$

$\Rightarrow (x^2-1)y_2 + xy_1 - n^2y = 0$ <u>1st part Proved</u>

Differentiating it n times with ~~respect to x~~, we get

$[y_2(x^2-1)]_n + [y_1x]_n - n^2[y]_n = 0$

$\Rightarrow y_{n+2}(x^2-1) + y_{n+1}\cdot n\cdot 2x + \frac{n(n-1)}{\cancel{2}}\cdot y_n\,\cancel{2}$

$+ y_{n+1}x + ny_n\cdot 1 - n^2y_n = 0$

$\Rightarrow (x^2-1)y_{n+2} + y_{n+1}(2n+1)x + y_n[n(n-1)+n-n^2]$

$\Rightarrow (x^2-1)y_{n+2} + (2n+1)xy_{n+1} + 0 = 0$

$\Rightarrow (x^2-1)y_{n+2} + (2n+1)xy_{n+1} = 0$

<u>2nd Part Proved</u>

2. If $I_n = \dfrac{d^n}{dx^n}(x^n \log x)$, prove that

(a) $I_n = n I_{n-1} + \lfloor n-1$

(b) $I_n = \lfloor n \left[\dfrac{1}{n} + \dfrac{1}{n-1} + \dfrac{1}{n-2} + \cdots + \dfrac{1}{2} + 1 + \log x \right]$

Soln Given that

$$I_n = \frac{d^n}{dx^n}(x^n \log x) \quad \text{---(1)}$$

$$\Rightarrow I_n = \frac{d^{n-1}}{dx^{n-1}}\left[\frac{d}{dx}(x^n \log x)\right]$$

$$\Rightarrow I_n = \frac{d^{n-1}}{dx^{n-1}}\left[n x^{n-1}\log x + x^{n-1}\right]$$

$$\Rightarrow I_n = n \frac{d^{n-1}}{dx^{n-1}}(x^{n-1}\log x) + \frac{d^{n-1}}{dx^{n-1}}(x^{n-1}) \quad \text{---(2)}$$

Replacing n by $n-1$ in (1), we get

$$I_{n-1} = \frac{d^{n-1}}{dx^{n-1}}(x^{n-1}\log x) \quad \text{---(3)}$$

Also, we know that $\dfrac{d^n}{dx^n}(x^n) = \lfloor n$

$$\Rightarrow \frac{d^{n-1}}{dx^{n-1}}(x^{n-1}) = \lfloor n-1 \quad \text{---(4)}$$

using (3) and (4) in (2), we get

$$I_n = n I_{n-1} + \lfloor n-1 \qquad \text{[Part (a) Proved]}$$

Dividing by $\lfloor n$, we get

$$\frac{I_n}{\lfloor n} = \frac{n I_{n-1}}{\lfloor n} + \frac{\lfloor n-1}{\lfloor n}$$

$$\Rightarrow \frac{I_n}{\lfloor n} = \frac{I_{n-1}}{\lfloor n-1} + \frac{1}{n} \quad \text{--- (5)}$$

Putting $n = n-1, n-2, \ldots, 3, 2$ successively, we get

$$\left.\begin{aligned} \frac{I_{n-1}}{\lfloor n-1} &= \frac{I_{n-2}}{\lfloor n-2} + \frac{1}{n-1} \\ \frac{I_{n-2}}{\lfloor n-2} &= \frac{I_{n-3}}{\lfloor n-3} + \frac{1}{n-2} \\ &\cdots\cdots \\ \frac{I_3}{\lfloor 3} &= \frac{I_2}{\lfloor 2} + \frac{1}{3} \\ \frac{I_2}{\lfloor 2} &= \frac{I_1}{\lfloor 1} + \frac{1}{2} \end{aligned}\right\} \text{--- (6)}$$

using (6) in (5), we have

$$\frac{I_n}{\lfloor n} = \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{1}{3} + \frac{1}{2} + \frac{I_1}{\lfloor 1} \quad \text{--- (7)}$$

Put $n = 1$ in (1), we get

$$I_1 = \frac{d}{dx}(x \log x) = \log x + 1$$

so (7) $\Rightarrow$ $$\frac{I_n}{\lfloor n} = \frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \cdots + \frac{1}{3} + \frac{1}{2} + 1 + \log x$$

$$\Rightarrow I_n = \lfloor n \left(\frac{1}{n} + \frac{1}{n-1} + \cdots + \frac{1}{2} + 1 + \log x \right]$$